An exactly solvable three-particle problem with three-body interaction

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Abstract

The energy spectrum of the three-particle Hamiltonian obtained by replacing the two-body trigonometric potential of the Sutherland problem by a three-body one of a similar form is derived. When expressed in appropriate variables, the corresponding wave functions are shown to be expressible in terms of Jack polynomials. The exact solvability of the problem with three-body interaction is explained by a hidden sl(3,$\mathbb{R}$) symmetry.

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In recent years, the Sutherland one-dimensional N-particle model [1] and its rational limit, the Calogero model [2], have received considerable attention in the literature. They are indeed relevant to several apparently disparate physical problems, such as fractional statistics and anyons [3], spin chain models [4], soliton wave propagation [5], two-dimensional non-perturbative quantum gravity and string theory [6], two-dimensional QCD [7], quantum chaotic systems and continuous matrix models [8].

Discovering new exactly solvable problems of a similar kind is of considerable interest and is therefore a topic of active research (see e.g. [9, 10]).

In the present paper, we shall present one such example, corresponding to a three-particle one-dimensional problem, wherein the particles are assumed to have equal masses, to move on an interval of length π/a, and to interact via a three-body trigonometric potential. We shall obtain the energy spectrum and the wave functions of the model when the particles are distinguishable or when they are indistinguishable and are either bosons or fermions. In addition, we shall prove that the model exact solvability can be explained by a hidden sl(3,$\mathbb{R}$) symmetry.

In units wherein $\hbar = 2m = 1$, the Hamiltonian of the problem can be written as

$$H = -\sum_{i=1}^{3} \partial_i^2 + 3fa^2 \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \csc^2\left(a(x_i + x_j - 2x_k)\right), \tag{1}$$

where x_i, $i = 1, 2, 3$, $0 \leq x_i \leq \pi/a$, denote the particle coordinates, $\partial_i \equiv \partial/\partial x_i$, and f is assumed to be such that $-1/4 < f \neq 0$. In the limit where $a \to 0$, the three-body trigonometric potential in (1) goes over into the three-body inverse square potential for particles moving on a line that was studied a long time ago by

Calogero and Marchioro [11], and, with an additional two-body harmonic potential, by Wolfes [12].

The Hamiltonian is invariant under translations of the centre-of-mass, whose coordinate will be denoted by $R = (x_1+x_2+x_3)/3$. In other words, H commutes with the total momentum $P = -i\sum_{i=1}^{3}\partial_i$, which may be simultaneously diagonalized. It will prove convenient to use two different systems of relative coordinates, namely $x_{ij} \equiv x_i - x_j$, $i \neq j$, and $y_{ij} \equiv x_i + x_j - 2x_k$, $i \neq j \neq k \neq i$, where in the latter, we suppressed index k as it is entirely determined by i and j.

Since for singular potentials crossing is not allowed, in the case of distinguishable particles the wave functions in different sectors of configuration space are disconnected, while for indistinguishable particles, they are related by a symmetry requirement. In the present case, the sector boundaries are determined by the vanishing of one of the variables y_{ij}. Since $y_{12} + y_{23} + y_{31} = 0$, in a given sector one of the variables y_{ij} must be of opposite sign to that of the remaining two. So there are altogether six sectors [11, 12], which may be labelled by an index $q = 0, 1, \ldots, 5$, as follows:

$$\begin{aligned}
q = 0 : (y_{12} > 0, y_{23} < 0, y_{31} < 0) &\equiv (x_{23} > 0, x_{31} < 0, |x_{12}| < \min(x_{23}, -x_{31}))\,, \\
q = 1 : (y_{12} > 0, y_{23} < 0, y_{31} > 0) &\equiv (x_{12} > 0, x_{31} < 0, |x_{23}| < \min(x_{12}, -x_{31}))\,, \\
q = 2 : (y_{12} < 0, y_{23} < 0, y_{31} > 0) &\equiv (x_{12} > 0, x_{23} < 0, |x_{31}| < \min(x_{12}, -x_{23}))\,, \\
q = 3 : (y_{12} < 0, y_{23} > 0, y_{31} > 0) &\equiv (x_{31} > 0, x_{23} < 0, |x_{12}| < \min(x_{31}, -x_{23}))\,, \\
q = 4 : (y_{12} < 0, y_{23} > 0, y_{31} < 0) &\equiv (x_{31} > 0, x_{12} < 0, |x_{23}| < \min(x_{31}, -x_{12}))\,, \\
q = 5 : (y_{12} > 0, y_{23} > 0, y_{31} < 0) &\equiv (x_{23} > 0, x_{12} < 0, |x_{31}| < \min(x_{23}, -x_{12}))\,.
\end{aligned}$$

Let us first assume that the particles are distinguishable and let us restrict the particle coordinates to a given sector of configuration space. By using the trigonometric identity

$$\sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \cot(ay_{ij}) \cot(ay_{jk}) = 2, \tag{2}$$

it is easy to show that the unnormalized ground-state wave function of Hamiltonian (1) is then given by

$$\psi_0(\boldsymbol{x}) = \prod_{\substack{i,j=1 \\ i\neq j}}^{3} |\sin(ay_{ij})|^{\lambda}, \tag{3}$$

and corresponds to a vanishing total momentum and to an energy eigenvalue $E_0 = 24a^2\lambda^2$, where $\lambda \equiv (1+\sqrt{1+4f})/2$ (implying that $f = \lambda(\lambda-1)$).

As usual in such a type of problem [1], the remaining solutions of the eigenvalue equations $H\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x})$, and $P\psi(\boldsymbol{x}) = p\psi(\boldsymbol{x})$ can be found by setting $\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x})\varphi(\boldsymbol{x})$. The function $\varphi(\boldsymbol{x})$ satisfies the equations $h\varphi(\boldsymbol{x}) = \epsilon\varphi(\boldsymbol{x})$, and $P\varphi(\boldsymbol{x}) = p\varphi(\boldsymbol{x})$, where $\epsilon = E - E_0$, and the gauge-transformed Hamiltonian $h \equiv (\psi_0(\boldsymbol{x}))^{-1}(H - E_0)\psi_0(\boldsymbol{x})$ can be written as

$$h = -\sum_{i=1}^{3} \partial_i^2 - \lambda a \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \cot\left(ay_{ij}\right)\left(\partial_i + \partial_j - 2\partial_k\right). \tag{4}$$

In terms of the new variables $z_i \equiv \exp\left(\frac{2}{3}ia(x_i - 2x_j + 4x_k)\right)$, where $(ijk) = (123)$, h and P become

$$h = 12a^2\left(\sum_i \left(z_i\partial_{z_i}\right)^2 + \lambda \sum_{\substack{i,j \\ i\neq j}} \frac{z_i + z_j}{z_i - z_j} z_i\partial_{z_i}\right) - \frac{8}{3}a^2\left(\sum_i z_i\partial_{z_i}\right)^2, \tag{5}$$

and

$$P = 2a\sum_i z_i\partial_{z_i}, \tag{6}$$

respectively. Equations (5) and (6) bear a resemblance to corresponding results for the Sutherland potential, with z_i defined in such a case by $z_i \equiv \exp(2iax_i)$. This hints at a possibility of expressing the simultaneous eigenfunctions of h and P in terms of Jack polynomials as in the case of the Sutherland potential [13].

By using Eq. (11) and theorems 3.1 and 5.1 of Ref. [14], it is indeed straightforward to prove that such simultaneous (unnormalized) eigenfunctions are given by

$$\varphi_{\{k\}}(\boldsymbol{x}) = \exp(6iaqR) J_{\{\mu\}}\left(\boldsymbol{z}; \lambda^{-1}\right), \tag{7}$$

and that there are no further eigenfunctions linearly independent from (7). Here $q \in \mathbb{R}$, and $J_{\{\mu\}}\left(\boldsymbol{z}; \lambda^{-1}\right)$ denotes the Jack (symmetric) polynomial in the variables z_i, $i = 1, 2, 3$, corresponding to the parameter λ^{-1}, and the partition $\{\mu\} = \{\mu_1 \mu_2\}$ into not more than two parts. The associated eigenvalues of h and P are

$$\epsilon_{\{k\}} = 4a^2 \left[3 \sum_i k_i^2 - \tfrac{2}{3} \left(\sum_i k_i\right)^2 - 6\lambda^2\right], \tag{8}$$

and

$$p_{\{k\}} = 2a \sum_i k_i = 2a \left(\sum_i \mu_i + 3q\right), \tag{9}$$

respectively. In Eqs. (7), (8), and (9), $\{k\} = \{k_1 k_2 k_3\}$, where k_1, k_2, and k_3 are defined by

$$k_1 = q - \lambda, \qquad k_2 = \mu_2 + q, \qquad k_3 = \mu_1 + q + \lambda. \tag{10}$$

The gauge-transformed Hamiltonian h can be separated into two parts, describing the centre-of-mass and relative motions respectively, $h = h^{cm} + h^{rel}$, where $h^{cm} = P^2/3$. As in the case of the Sutherland potential [15], it is advantageous to

write the relative Hamiltonian h^{rel} in terms of new variables. By setting

$$v_i \equiv \exp(-2iax_{jk}) = z_i \exp(-2iaR) \tag{11}$$

for $(ijk) = (123)$, and

$$\zeta_1 \equiv \sum_i v_i, \qquad \zeta_2 \equiv \sum_{i<j} v_i v_j = \sum_i v_i^{-1}, \tag{12}$$

one finds

$$h^{rel} = 8a^2 \Big[\left(\zeta_1^2 - 3\zeta_2\right) \partial^2_{\zeta_1} + (\zeta_1\zeta_2 - 9)\partial^2_{\zeta_1\zeta_2} + \left(\zeta_2^2 - 3\zeta_1\right) \partial^2_{\zeta_2} + (3\lambda + 1)\left(\zeta_1 \partial_{\zeta_1} + \zeta_2 \partial_{\zeta_2}\right) \Big]. \tag{13}$$

The eigenfunctions and eigenvalues of h can be similarly separated into centre-of-mass and relative contributions,

$$\begin{aligned}
\varphi_{\{k\}}(\boldsymbol{x}) &= \varphi^{cm}_{\{k\}}(\boldsymbol{x})\varphi^{rel}_{\{\mu\}}(\boldsymbol{x}), &(14)\\
\varphi^{cm}_{\{k\}}(\boldsymbol{x}) &= \exp\left(ip_{\{k\}}R\right) = \exp\left[2ia\left(\sum_i k_i\right)R\right], &(15)\\
\varphi^{rel}_{\{\mu\}}(\boldsymbol{x}) &= J_{\{\mu\}}\left(\boldsymbol{v}; \lambda^{-1}\right) = P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right), &(16)
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_{\{k\}} &= \epsilon^{cm}_{\{k\}} + \epsilon^{rel}_{\{\mu\}}, &(17)\\
\epsilon^{cm}_{\{k\}} &= \frac{1}{3}p^2_{\{k\}} = \frac{4}{3}a^2\left(\sum_i k_i\right)^2, &(18)\\
\epsilon^{rel}_{\{\mu\}} &= 4a^2\left[3\sum_i k_i^2 - \left(\sum_i k_i\right)^2 - 6\lambda^2\right]. &(19)
\end{aligned}$$

In Eq. (16), $P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right)$ is the polynomial in ζ_1 and ζ_2, characterized by the parameter λ^{-1} and the partition $\{\mu\} = \{\mu_1\mu_2\}$, that is obtained from the corresponding Jack polynomial in the variables v_i by making the change of variables (12). In Eq. (19),

the relative-motion energy $\epsilon^{rel}_{\{\mu\}}$ actually depends only upon the partition $\{\mu\}$, and not upon q, and may be written as

$$\epsilon^{rel}_{\{\mu\}} = 8a^2 \left(\mu_1^2 - \mu_1\mu_2 + \mu_2^2 + 3\lambda\mu_1\right), \tag{20}$$

so that $P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right)$ satisfies the eigenvalue equation

$$h^{rel} P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right) = \epsilon^{rel}_{\{\mu\}} P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right). \tag{21}$$

For instance, for $\{\mu\} = \{2\}$, one finds $J_{\{2\}}\left(\boldsymbol{z}; \lambda^{-1}\right) = \sum_i z_i^2 + 2\lambda(\lambda+1)^{-1} \sum_{i<j} z_i z_j$, and $P_{\{2\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right) = \zeta_1^2 - 2(\lambda+1)^{-1}\zeta_2$. For general $\{\mu\}$, one can show that $P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right)$ belongs to the space $V_{\mu_1}(\boldsymbol{\zeta})$, where $V_n(\boldsymbol{\zeta})$, $n \in \mathbb{N}$, is defined as the space of polynomials in ζ_1 and ζ_2 that are of degree less than or equal to n (hence, $\dim V_n = (n+1)(n+2)/2$).

Let us now consider the full configuration space for distinguishable particles. Let $\varphi^{(q)rel}_{\{\mu\}}(\boldsymbol{x})$ denote the function coinciding with function (16) in sector q, and vanishing in the remaining five sectors. It is obvious that the six wave functions $\varphi^{(q)rel}_{\{\mu\}}(\boldsymbol{x})$, $q = 0, 1, \ldots, 5$, corresponding to a given partition $\{\mu\}$, are associated with the same eigenvalue $\epsilon^{rel}_{\{\mu\}}$ of h^{rel}. In addition, we note from (20) that $\epsilon^{rel}_{\{\mu_1\mu_2\}} = \epsilon^{rel}_{\{\mu_1,\mu_1-\mu_2\}}$. Hence, for generic (i.e. irrational) λ values, the relative energy spectrum levels characterized by any partition $\{\mu_1\mu_2\}$ for which $\mu_1 > 2\mu_2$ have a twelvefold degeneracy, whereas those for which $\mu_1 = 2\mu_2$ have only a sixfold degeneracy.

Such degeneracies can be explained by considering the symmetry group of h^{rel}, which is a group of order 12, obtained by combining the six particle permutations with the identity and the parity transformation $\Pi : x_{ij} \to -x_{ij}$. These transformations act both on the variables, and on their domains, i.e., the configuration space sectors.

The variables ζ_1 and ζ_2 remain invariant under any even permutation, but are interchanged under any odd permutation or the parity transformation. From Eq. (16) and the properties of Jack polynomials, it results that

$$P_{\{\mu_1\mu_2\}}\left(\zeta_2, \zeta_1; \lambda^{-1}\right) = P_{\{\mu_1,\mu_1-\mu_2\}}\left(\zeta_1, \zeta_2; \lambda^{-1}\right), \tag{22}$$

showing that the interchange of ζ_1 and ζ_2 is equivalent to the replacement of $\{\mu_1\mu_2\}$ by $\{\mu_1, \mu_1 - \mu_2\}$.

Each q-sector is invariant under one odd permutation, but under the remaining permutations, is changed into the sectors for which q has the same parity. For instance, $q = 0 \to q = 0$ under (12), $q = 0 \to q = 2$ under (23) or (132), and $q = 0 \to q = 4$ under (31) or (123). The parity transformation, on the other hand, mixes the even-q-sectors with the odd ones, e.g., $q = 0 \to q = 3$.

By combining these results, it is now clear that the functions $\varphi^{(q)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x})$, and $\varphi^{(q)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x})$ for $q = 0, 1, \ldots, 5$, and $\mu_1 > 2\mu_2$, or $\varphi^{(q)rel}_{\{\mu_2,2\mu_2\}}(\boldsymbol{x})$ for $q = 0, 1, \ldots, 5$, span the representation space of some irreducible representation of the relative Hamiltonian symmetry group, as it should be.

Let us next consider the case of indistinguishable particles, either bosons or fermions. The only allowed wave functions are then completely symmetrical or antisymmetrical functions, respectively. The previous discussion shows that they are given by

$$\begin{aligned}\varphi^{(\pm)(e)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) \quad = \quad & \varphi^{(0)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) + \varphi^{(2)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) + \varphi^{(4)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) \\ & \pm \left[\varphi^{(0)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x}) + \varphi^{(2)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x}) + \varphi^{(4)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x})\right], \end{aligned} \tag{23}$$

and

$$\varphi^{(\pm)(o)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) = \varphi^{(1)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) + \varphi^{(3)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) + \varphi^{(5)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) \pm \left[\varphi^{(1)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x}) + \varphi^{(3)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x}) + \varphi^{(5)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x})\right], \quad (24)$$

where the upper (resp. lower) signs correspond to bosons (resp. fermions). The relative energy spectrum levels are characterized by the partitions $\{\mu_1\mu_2\}$, such that $\mu_1 \geq 2\mu_2$ for bosons, or $\mu_1 > 2\mu_2$ for fermions, and they have a residual twofold degeneracy coming from the invariance of h^{rel} under Π. The linear combinations $\varphi^{(\pm)(e)rel}_{\{\mu_1\mu_2\}} \pm \varphi^{(\pm)(o)rel}_{\{\mu_1\mu_2\}}$, and $\varphi^{(\pm)(e)rel}_{\{\mu_1\mu_2\}} \mp \varphi^{(\pm)(o)rel}_{\{\mu_1\mu_2\}}$ have a given parity, even or odd, respectively.

The relative energy spectrum (in appropriate units) and its degeneracies are the same as for three particles on an interval of length π/a, interacting via the two-body potential $\kappa(\kappa-1)a^2\sum_{i\neq j}\csc^2(ax_{ij})$ whenever the particles are distinguishable, but they are distinct for indistinguishable particles. This is due to the fact that both the configuration space sectors, and the variables the relative wave functions depend upon have different transformation properties under permutations for the two potentials.

For the two-body potential, the configuration space sectors may be labelled by an index $p = 0, 1, \ldots, 5$, and defined as follows: $p=0$: $(x_{12} > 0, x_{23} > 0, x_{31} < 0)$, $p = 1$: $(x_{12} > 0, x_{23} < 0, x_{31} < 0)$, $p = 2$: $(x_{12} > 0, x_{23} < 0, x_{31} > 0)$, $p = 3$: $(x_{12} < 0, x_{23} < 0, x_{31} > 0)$, $p = 4$: $(x_{12} < 0, x_{23} > 0, x_{31} > 0)$, $p = 5$: $(x_{12} < 0, x_{23} > 0, x_{31} < 0)$ [1, 11, 12]. Any one of them is transformed into the five remaining sectors under permutations. On the other hand, the $p=0$ and $p=3$, $p=1$ and $p=4$, $p=2$ and $p=5$ sectors are interchanged under Π.

The relative wave functions in a given sector now assume the form [15]

$$\varphi^{rel}_{\{\mu\}}(\boldsymbol{x}) = J_{\{\mu\}}\left(\boldsymbol{w};\kappa^{-1}\right) = P_{\{\mu\}}\left(\boldsymbol{\eta};\kappa^{-1}\right), \tag{25}$$

where $w_i \equiv \exp(-2iay_{jk}/3)$ for $(ijk) = (123)$, and $\eta_1 \equiv \sum_i w_i$, $\eta_2 \equiv \sum_{i<j} w_i w_j = \sum_i w_i^{-1}$. From their definition, it results that the variables η_1 and η_2 remain invariant under permutations, but are interchanged under Π.

For bosons or fermions, the relative wave functions for the two-body potential may therefore be written as

$$\varphi^{(\pm)rel}_{\{\mu\}}(\boldsymbol{x}) = \varphi^{(0)rel}_{\{\mu\}}(\boldsymbol{x}) + \varphi^{(2)rel}_{\{\mu\}}(\boldsymbol{x}) + \varphi^{(4)rel}_{\{\mu\}}(\boldsymbol{x}) \pm \left[\varphi^{(1)rel}_{\{\mu\}}(\boldsymbol{x}) + \varphi^{(3)rel}_{\{\mu\}}(\boldsymbol{x}) + \varphi^{(5)rel}_{\{\mu\}}(\boldsymbol{x})\right], \tag{26}$$

where $\varphi^{(p)rel}_{\{\mu\}}(\boldsymbol{x})$ denotes as before the function coinciding with function (25) in sector p, and vanishing in the remaining five sectors. In Eq. (26), $\{\mu\}$ runs over all partitions into not more than two parts. The relative energy spectrum levels are characterized by $\{\mu_1\mu_2\}$, where $\mu_1 \ge 2\mu_2$ for both bosons and fermions. Those with $\mu_1 > 2\mu_2$ have a twofold degeneracy coming from the invariance of h^{rel} under Π. The corresponding even and odd wave functions are given by $\varphi^{(\pm)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) \pm \varphi^{(\pm)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x})$, and $\varphi^{(\pm)rel}_{\{\mu_1\mu_2\}}(\boldsymbol{x}) \mp \varphi^{(\pm)rel}_{\{\mu_1,\mu_1-\mu_2\}}(\boldsymbol{x})$, respectively. In contrast, the levels with $\mu_1 = 2\mu_2$ are not degenerate, the corresponding wave function $\varphi^{(+)rel}_{\{\mu_2,2\mu_2\}}(\boldsymbol{x})$ (resp. $\varphi^{(-)rel}_{\{\mu_2,2\mu_2\}}(\boldsymbol{x})$) being even (resp. odd).

As a final point, we shall now proceed to show that the exact solvability of H, defined in Eq. (1), or equivalently of h^{rel}, defined in Eq. (13), is due to a hidden sl(3,$\mathbb{R}$) symmetry. For such purpose, let us consider the operators E_{ij}, i, $j = 1, 2, 3$,

defined by

$$\begin{aligned}
E_{11} &= \zeta_1\partial_{\zeta_1}, \qquad E_{22} = \zeta_2\partial_{\zeta_2}, \qquad E_{33} = n - \zeta_1\partial_{\zeta_1} - \zeta_2\partial_{\zeta_2}, \\
E_{31} &= \partial_{\zeta_1}, \qquad E_{32} = \partial_{\zeta_2}, \qquad E_{21} = \zeta_2\partial_{\zeta_1}, \qquad E_{12} = \zeta_1\partial_{\zeta_2}, \\
E_{13} &= n\zeta_1 - \zeta_1^2\partial_{\zeta_1} - \zeta_1\zeta_2\partial_{\zeta_2}, \qquad E_{23} = n\zeta_2 - \zeta_1\zeta_2\partial_{\zeta_1} - \zeta_2^2\partial_{\zeta_2}, \qquad (27)
\end{aligned}$$

where n may take any real value. It is clear [15, 16] that they fulfil the gl(3,$\mathbb{R}$) commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}. \qquad (28)$$

Since the linear combination $\sum_i E_{ii}$ reduces to a constant, Eq. (27) actually provides a representation of the traceless part sl(3,$\mathbb{R}$) of gl(3,$\mathbb{R}$), acting on the space of functions in ζ_1 and ζ_2. Whenever n is a non-negative integer, such a representation reduces to a finite-dimensional one on the space $V_n(\boldsymbol{\zeta})$ of polynomials in ζ_1 and ζ_2 that are of degree less than or equal to n.

It is now straightforward to prove that h^{rel} belongs to the enveloping algebra of sl(3,$\mathbb{R}$). It can indeed be rewritten as the following quadratic combination of the E_{ij}'s,

$$h^{rel} = 8a^2 \left[E_{11}^2 + E_{11}E_{22} + E_{22}^2 - 3E_{12}E_{32} - 3E_{21}E_{31} - 9E_{31}E_{32} + 3\lambda\left(E_{11} + E_{22}\right)\right]. \qquad (29)$$

Such an expression is valid for any real n value. Hence, the operator h^{rel} possesses infinitely many finite-dimensional invariant subspaces $V_n(\boldsymbol{\zeta})$, $n = 0, 1, 2, \ldots$, and, correspondingly, preserves an infinite flag of spaces, $V_0(\boldsymbol{\zeta}) \subset V_1(\boldsymbol{\zeta}) \subset V_2(\boldsymbol{\zeta}) \subset \ldots$. In the basis wherein all spaces $V_n(\boldsymbol{\zeta})$ are naturally defined, the matrix representing h^{rel} is therefore triangular, so that h^{rel} is exactly solvable [16].

In conclusion, we did prove in the present paper that various results valid for the N-particle Sutherland problem can be extended to the three-particle problem, wherein the Sutherland two-body trigonometric potential is replaced by a three-body potential of a similar form. In particular, the wave functions can still be expressed in terms of Jack polynomials [13], such a property being related with the existence of a hidden sl(3,$\mathbb{R}$) symmetry [15].

Although the wave functions of the problem with two-body interaction and of the present one look similar when expressed in appropriate variables, they are actually rather different when rewritten in terms of the particle coordinates x_i, $i = 1, 2, 3$. As a consequence, for indistinguishable particles for which permutations of the x_i's play a crucial role, the energy spectra of the two problems are distinct. Comparing other properties of the three-particle system for the two problems might be an interesting question for future study.

It is not clear yet whether the three-particle problem with both two- and three-body trigonometric potentials is exactly solvable as its rational limit [11, 12]. Investigating this point would be of interest too.

One should also note that the N-particle problem with three-body trigonometric interaction is unlikely to be solvable as the three-particle one since the latter is connected with the exceptional Lie algebra G_2 [17]. This contrasts with the case of the N-particle problem with two-body trigonometric interaction, which is related to the Lie algebra A_{N-1} for any $N = 2, 3, \ldots$.

As for the Sutherland potential [13], one might also consider a generalized spin-dependent Hamiltonian for particles with internal degrees of freedom. In such a

case, use could be made of the three first-order differential-difference operators that were recently introduced in connection with the Weyl group D_6 of G_2 [18, 19].

References

[1] B. Sutherland, Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972); Phys. Rev. Lett. **34**, 1083 (1975).

[2] F. Calogero, J. Math. Phys. **10**, 2191 (1969); **10**, 2197 (1969); **12**, 419 (1971).

[3] J. M. Leinaas and J. Myrheim, Phys. Rev. B **37**, 9286 (1988); A. P. Polychronakos, Nucl. Phys. B **324**, 597 (1989); Phys. Lett. B **264**, 362 (1991); F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[4] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); **66**, 1529 (1991); B. S. Shastry, *ibid.* **60**, 639 (1988).

[5] H. H. Chen, Y. C. Lee, and N. R. Pereira, Phys. Fluids **22**, 187 (1979).

[6] V. A. Kazakov, in *Random Surfaces and Quantum Gravity (Cargese Lectures, 1990)*, edited by O. Alvarez *et al* (Plenum, New York, 1991).

[7] J. A. Minahan and A. P. Polychronakos, Phys. Lett. B **326**, 288 (1994).

[8] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993); **72**, 64 (1994).

[9] T. Yamamoto, Phys. Lett. A **208**, 293 (1995).

[10] A. Khare, J. Phys. A **29**, L45 (1996).

[11] F. Calogero and C. Marchioro, J. Math. Phys. **15**, 1425 (1974).

[12] J. Wolfes, J. Math. Phys. **15**, 1420 (1974).

[13] F. Lesage, V. Pasquier, and D. Serban, Nucl. Phys. B **435**, 585 (1995).

[14] R. P. Stanley, Adv. Math. **77**, 76 (1989).

[15] W. Rühl and A. Turbiner, Mod. Phys. Lett. A **10**, 2213 (1995).

[16] A. V. Turbiner, in *Lie Algebras, Cohomologies and New Findings in Quantum Mechanics (Contemporary Mathematics 160)*, edited by N. Kamran and P. Olver (AMS, Providence, RI, 1994) p. 263.

[17] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **94**, 313 (1983).

[18] C. Quesne, Mod. Phys. Lett. A **10**, 1323 (1995).

[19] C. Quesne, Europhys. Lett. **35**, 407 (1996).